The diffusion of smoke from a continuous elevated point-source into a turbulent atmosphere

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SUMMARY

The problem is to determine the downwind concentration of non-buoyant smoke emitted from a continuous elevated point-source in a turbulent airstream. The velocity and eddy diffusivity coefficients are represented by related powers of the height above ground, and are independent of the position of the source. Exact solutions are obtained for the zero and second moments of the concentration distribution along lines lying in the cross-wind direction at ground level. In special cases, these moments may also be determined along lines at general height. In one such case the concentration is determined exactly (rather than just the two moments) and it is found that the cross-wind distribution always has a Gaussian form. If it is assumed that in all cases the cross-wind profile is Gaussian, a formulation for the concentration can be given purely in terms of the known zero and second moments. When the source is moving with constant velocity across the wind, the first moment as well as the zero and second moments is exactly determined, and under a similar assumption a formula for the concentration is found.

1. INTRODUCTION

A new approach is used in tackling the problem of the determination of the concentration of non-buoyant smoke downstream of grounded and elevated point-sources in a statistically-steady turbulent airstream.

When the mean velocity is represented by a power of the height,

$$u = u_0(z+h)^\alpha, \tag{1.1}$$

(z is the height relative to the source height h; see figure 1) and the eddy diffusivity coefficient K is related to the velocity by the usual conjugate power relation based on Reynolds analogy, an exact solution for the concentration due to an elevated line-source is found for general values of the velocity profile parameter α. This solution, together with a further exact solution for the 'spread' of the plume from a point-source, again for general α, enables an expression for the ground level concentration, due to an elevated point-source, to be found on the basis of the single assumption that this concentration has a Gaussian profile in the horizontal cross-wind y-direction. The restriction, that this expression is for the concentration at ground level only, rather than at all heights, is removed when the source itself is at ground level, no matter what value α may take, and also when the source is elevated provided that α is zero or one-half.

The nature of these solutions and their success throws doubt on the theoretical basis of some previous solutions and on their range of applicability.

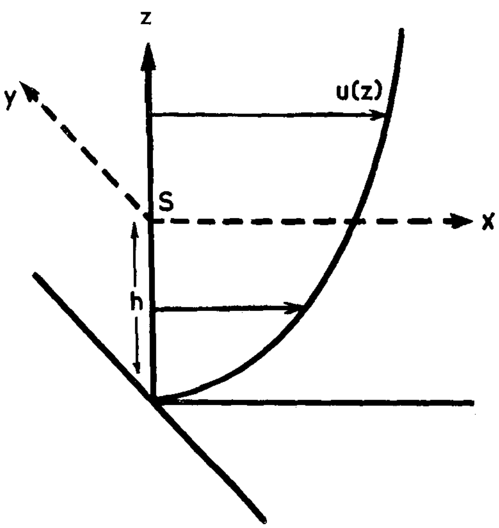

Figure 1. The system of axes in relation to the position of the source, elevation h, and the unidirectional velocity field $u(z)$.

Before the mathematical analysis is given, it is desirable to elucidate and to add to the above statements. In particular, it will be necessary to discuss the nature of the eddy diffusivity coefficients, as some forms suggested fairly recently by other workers are fundamentally different in concept. How these other forms arose and developed is perhaps best seen in their historical setting.

The equation of diffusion is

$$u(z)\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right), \tag{1.2}$$

where x is the direction of the mean stream and C stands for the concentration. The diffusion in the direction of the mean stream is neglected, on the usual 'boundary-layer' approximations, by comparison with the other terms in the equation. It remains to specify the form of $u(z)$, K_z and K_y. Following Calder and others, the velocity is given by (1.1), which for problems of this type is a sufficiently good representation and is preferable on grounds of mathematical simplicity to more sophisticated profiles, such as the logarithmic profile based on Nikuradse's relation, or Deacon's law. The parameter α in equation (1.1) is dependent on the stability of the air: in neutral stability it is normally about $\frac{1}{7}$, and it increases as the stability increases. As we have stated, the form of the diffusivity coefficients is in dispute but it is true that there has been fairly general support for supposing that K_z, the coefficient representing the vertical diffusion process, satisfies the Reynolds analogy; that is, K_z is proportional to the coefficient of vertical momentum transfer and thus (under conditions of zero horizontal pressure gradient) satisfies the relationship that states that the shearing stress is constant with height:

$$K_z \frac{\partial u}{\partial z} = \text{constant}. \tag{1.3}$$

Thus if u satisfies (1.1) then K_z also obeys a simple power law

$$K_z = K_0(z+h)^{1-\alpha}. \tag{1.4}$$

At about 100 metres, the turbulence is almost perfectly isotropic; and even below this level the degree of isotropy seems sufficiently high to warrant letting K_y, as Sutton (1953) has suggested, vary in an identical way with K_z, namely,

$$K_y = K_0(z+h)^{1-\alpha}. \tag{1.5}$$

This assumption may be tested by the validity of the solutions derived with it. Under certain circumstances it is conceivable that a more general form for K_y would be appropriate. By taking

$$K_y = K_1(z+h)^{1-\alpha+\mu} \tag{1.6}$$

the solutions may still be found for sources at ground level and also for elevated sources when only the ground level concentrations are required. Thus, although we expect μ to be zero, its presence in the solutions will serve to indicate the influence of K_y.

The physical significance of these forms will be discussed at the end of the introduction in the comparison with previous work in this field.

In tackling the three-dimensional point-source problem it is, first of all, desirable to try to find any exact solutions of equations (1.1), (1.2), (1.4) and (1.5). Section 3 deals with the exact solution for the elevated point-source when $\alpha = \frac{1}{2}$. In this particular case it is possible to separate (in the mathematical sense) the variables y and z in the diffusion equation to give two separate differential equations, one with independent variables x and z, the other with y and x. The first equation is in fact the equation for a line source at the same x and z as the point-source, and the second equation gives the behaviour of the concentration profile with respect to y as Gaussian. It is therefore necessary to have the exact solution of the two-dimensional line-source problem. As will be seen below this solution is required not only for $\alpha = \frac{1}{2}$ but also for general values of α.

In extending the work to general α, for which an exact solution could not be obtained, it seemed desirable to accumulate first as much exact knowledge about the point-source plume as possible. With this in mind, the approach consists in determining exact solutions for two functions of the point-source concentration, namely

$$C_0 = \int_{-\infty}^{\infty} C\,dy \tag{1.7}$$

and
$$C_2 = \int_{-\infty}^{\infty} y^2 C\, dy, \tag{1.8}$$
as functions of x and z. The advantages of forming the differential equations for these two functions are that they are now two-dimensional, the boundary conditions are known, and their solutions represent two of the most important features of the complete solution C. The first function C_0 gives the total amount of concentration on any transverse line x = constant, z = constant; and, moreover, since integration of equation (1.2) with respect to y shows that C_0 is the solution of the equivalent line-source equation, it is of primary importance to solve the line-source problem for this reason, apart from the fact that it is of course an interesting problem in its own right. It is worth while noting that the term representing the lateral diffusion, and thus K_y, does not enter into the equation for C_0. The solution for the general elevated source is derived in §2 and is discussed below.

The second function is the second moment of the transverse concentration profile and thus C_2/C_0 is a measure of the 'spread' in the y-direction.

Now, since we noted that in the exact solution for $\alpha = \frac{1}{2}$ the concentration profiles had Gaussian transverse cross-sections, it is reasonable to use the exact solutions C_0 and C_2 to formulate for general α an expression for C which satisfies all the boundary conditions and which is based on the single hypothesis of a Gaussian transverse cross-section. Thus
$$C = X(x,z)e^{-y^2/f(x,z)}, \tag{1.9}$$
where $X(x,z)$ and $f(x,z)$ can be expressed in terms of C_0 and C_2 as follows (using (1.7) and (1.8)):
$$f = 2C_2/C_0 \quad \text{and} \quad X = C_0\sqrt{(C_0/2\pi C_2)}. \tag{1.10}$$
The choice of the Gaussian form (1.9) is supported by the fact that, since the diffusion equation contains only the single derivative with respect to y, $K_y\,\partial^2 C/\partial y^2$, with K_y independent of y, the diffusion equation effectively represents a process of simple diffusion in the lateral direction, a process with which Gaussian profiles are usually associated.

Since it is the spread which essentially determines the rate of decay of C for large x, it is to be expected that the expression (1.9) will have the additional property that it has the correct asymptotic behaviour. The general elevated line-source problem has as a particular solution the case of zero source-elevation h, which was historically one of the first to be found. This solution (5.1) which has received ample experimental verification is, in fact, included in the solution for general source elevation that is derived in §2 by the use of operational methods. This latter solution involves a Bessel function but is nevertheless readily computed (especially if all that is required is the concentration C_0 at ground level $z = -h$). We have
$$C_0(x,z) = \frac{Q}{(2\alpha+1)}\frac{(zh+h^2)^{\alpha/2}}{K_0 x}\exp\left[-\frac{u_0(z+h)^{1+2\alpha}+u_0 h^{1+2\alpha}}{K_0(2\alpha+1)^2 x}\right]\times$$
$$\times I_{-\alpha/(1+2\alpha)}\left[\frac{2u_0(zh+h^2)^{(1+2\alpha)/2}}{K_0(2\alpha+1)^2 x}\right], \tag{1.11}$$

which in the case of $\alpha = \frac{1}{7}$ has the shape shown in figure 2 for $y = 0$, $z = -h$. The curves for other values of α are similar in their general shape although the precise details will of course vary. It is interesting to note that as one travels in the direction of increasing x the jump in the concentration from zero to an appreciable value takes place very rapidly.

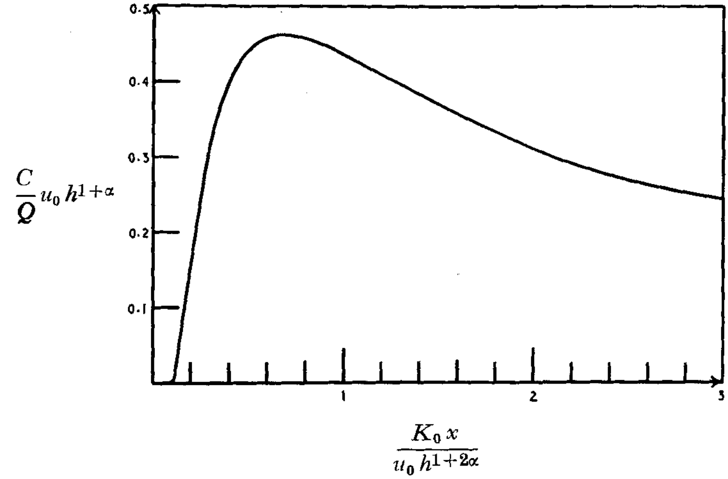

Figure 2. The concentration at ground level, as a function of the distance downstream due to an elevated line source when $\alpha = \frac{1}{7}$.

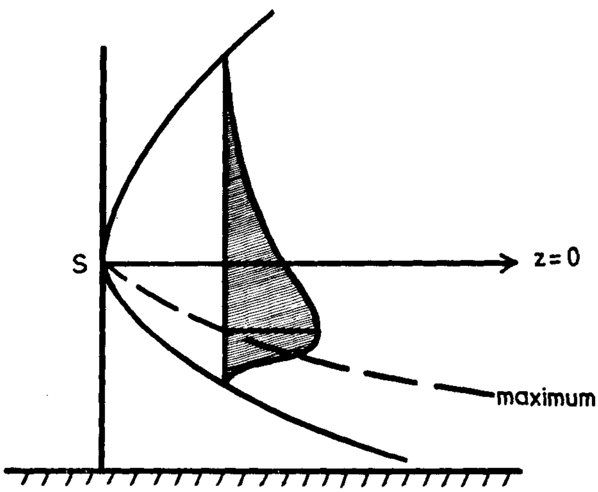

Figure 3. Diagrammatic representation of the distribution of concentration along the vertical near the source.

The solution has the same asymptotic behaviour for large x as the verified particular solution for h zero; for example, in conditions of neutral stability ($\alpha = \frac{1}{7}$) the concentration behaves like the $(-\frac{8}{9})$th power of x. This ultimate similarity for all h is to be expected on realizing that the smoke distribution 'forgets' its initial distribution as it is swept downstream.

Near the source the concentration is asymmetrically distributed about the plane $z = 0$ (this is diagrammatically represented in figure 3). This asymmetry may be accounted for by the combined effect of the variation of K and u with height in the following way. As the vertical diffusive processes increase with height, the smoke is distributed along the vertical more rapidly above the source than below it, so that the concentration profile near $x = 0$ becomes relatively distended above the source. Initially, then, the concentration is greater at, say, $z = h$ than at $z = -h$ (as figure 10 shows). The velocity shear also accentuates the 'asymmetry' effect; if, for a moment, one neglects diffusion and imagines that a steady source injects smoke at a given rate into a steady stream then it is clear that the density of the smoke particles in that stream would be inversely proportional to its speed. Thus the effect of the velocity shear is to increase the concentration below the source, where the stream speed is lower, and to decrease it above the source, remembering however that the actual amount in any layer is governed largely by the diffusion process.

The differential equation for C_2 is somewhat more complicated than that for C_0, owing to the addition of an extra term depending on the functional nature of K_y and on C_0 itself. For zero source-elevation h, the method of solution is fairly straightforward and consists of expressing C_2 in the form

$$C_2 = x^{\lambda}\overline{C}(\eta), \qquad \eta = \frac{u_0\, z^{1+2\alpha}}{(1+2\alpha)^2 K_0\, x}.$$

By solving for $\overline{C}(\eta)$ and applying the appropriate boundary conditions, C_2 is finally expressed in terms of a confluent hypergeometric function and an allied function V with similarly rapid convergence properties:

$$\left.\begin{aligned} V(b;a;\eta) &= \sum_{r=0}^{\infty} \frac{(2b+r-1)!}{(b+r)!\,(b+a+r-1)!}\eta^r, \qquad b = \frac{2}{1+2\alpha}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad a = \frac{1+\alpha}{1+2\alpha}; \\ C_2 &= 2Q\sqrt{\left(\frac{K_1}{K_0}\right)}\left(\frac{2}{b}\right)^{(3b-4)/2} \frac{K_0^{b-a}}{u_0^{b-a+1}} \frac{(b-1)!\,(b+a-2)!}{(a-1)!\,(2b-1)!} x^{b-a} \times \\ &\qquad \times e^{-\eta}\left[\frac{(b-1)!}{(a-1)!}\,{}_1F_1(b;a;\eta) - \eta^b V(b;a;\eta)\right]. \end{aligned}\right\} \quad (1.12)$$

The behaviour of C_2 is chiefly governed by the term $e^{-\eta}$ for small x, and by the term x^{b-a} for large x. Thus the general shape is as shown in figure 4. It is found that the solution of the differential equation is simplified when α takes the values 0, $\frac{1}{3}$, $\frac{1}{2}$ and 1.

The problem of the elevated source does not meet with quite the same general success, although C_2 may be directly determined at all levels for $\alpha = 0$ (as well as for $\alpha = \frac{1}{2}$, for which the exact solution has already been quoted) and at ground level for $\alpha = \frac{1}{3}$ and 1. (It seems reasonable to suppose that the solutions for these two cases could likewise be extended to all levels if so desired, but at present the actual method is not clear.) The difficulty for general α is not theoretical but rather a matter of being stranded with an unpleasant indefinite integral (6·6) to evaluate, and this it is possible to do in terms of simple functions only in the above four cases. As in the case of the elevated line-source, the solutions are obtained by the use of operational methods.

However, for most purposes the values of the concentration are required only at ground level, and this is certainly true in problems of atmospheric pollution. This greatly simplifies the problem since the concentration at ground level downwind of a point- or a line-source elevated at height h above ground is identical with the concentration at the same x and y, but at height h, due to an exactly similar source at ground level. This property of reciprocity has been noticed by Bosanquet & Pearson (1936) in the case of the line-source when $\alpha = 0$, and in the obvious case of u = constant,

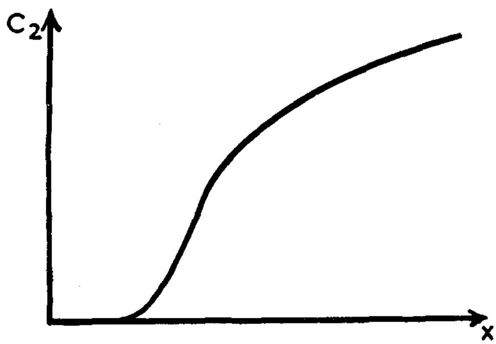

Figure 4. Diagrammatic representation of the variation of the 'spread' C_2 with distance downstream.

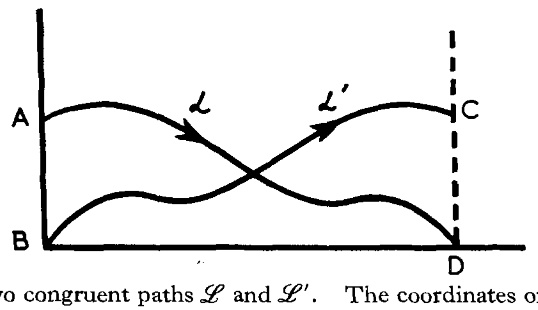

Figure 5. The two congruent paths $\mathscr{L}$ and $\mathscr{L}'$. The coordinates of the points are: $A = (0, 0, h)$, $B = (0, 0, 0)$, $C = (x, y, h)$, $D = (x, y, 0)$.

K = constant. They realized that some general theorem of the nature of the one given here might be true. This reciprocal theorem is rigorously proved in § 5 and is a consequence of the fact that the solutions of the problem are Green's functions. Physically the explanation of this theorem is not so straightforward; if, however, one assumes that the distribution of perturbation velocities at any point in the field is symmetrical, then the probability that a particle leaving the elevated source follows a certain path $\mathscr{L}$ to a point $(x, y, 0)$ at ground level is the same as the probability that a particle leaving the same source now at ground level will follow the congruent path $\mathscr{L}'$ to the point (x, y, h), since both particles pass through corresponding points and conditions, only in opposite order. Integration of all such possible paths recovers the theorem. Hence the concentration at ground level due to the elevated source may be determined from (1.12) by putting $z = h$, that is $\eta = u_0 h^{1+2\alpha}/(1+2\alpha)^2 K_0 x$.

The results are plotted for $\alpha = 0, \frac{1}{7}, \frac{1}{3}, \frac{1}{2}$ and 1 in § 7, and a comparison between them is made. From these solutions, interpolation curves for the height and distance downstream of the maximum concentration are deduced and it is noted that the greatest variation of these factors occurs for small α.

In the final section, § 8, a simple extension of § 4 is investigated. The problem is made more general by allowing the source to move (with the application to things like smoke-screen layers in mind) with velocity constant in the y-direction. Again C_0 and C_2 are found, relative to axes moving with the source. Furthermore, the function

$$C_1 = \int_{-\infty}^{\infty} yC\,dy \tag{1.13}$$

is determined. The function $C_1/C_0 = Y(x, z)$ is a measure of the lateral displacement of the centre of the plume due to the motion of the source. For large x, this displacement is proportional to $x^{(1+\alpha)/(1+2\alpha)}$, which implies that, for $\alpha > 0$, the mean velocity of a group of smoke particles is tending to increase with time as the group is swept downstream due to the general deepening of the plume and the increase of velocity with height.

To conclude this introduction, we discuss the nature of the eddy diffusivity in the light of previous solutions and formulations. Naturally, the form chosen for K_y has an important influence on the character of the solution; in particular, on the rate of decrease of concentration for large x. Briefly this is because, since this rate of decay is asymptotically the same at all levels, the overall decrease, which is governed by how quickly it spreads out laterally, is the greater the faster K_y increases with height. For this reason the elementary solution corresponding to u = constant, $K_y = K_z = K$ = constant,

$$C = \frac{Q}{4\pi x K}\left\{\exp\left[-\frac{(y^2+(z-h)^2)u}{4Kx}\right]+\exp\left[-\frac{(y^2+(z+h)^2)u}{4Kx}\right]\right\}, \tag{1.14}$$

is not applicable in the atmosphere. (In (1.14) $z = 0$ is ground level, h the source elevation.) We note that this solution gives a rate of decrease like x^{-1} whereas experiments (Sutton 1947) show that this index of x should be about $-1{\cdot}76$ (cf. equations (1.12), (1.11) which give an index $-1{\cdot}67$ for $\alpha = \frac{1}{7}$).

Similar objections (Sutton 1953) have been raised against the solutions due to Davies (1950) for which u and K_z satisfy (1.1) and (1.4) respectively and K_y obeys the power law relation

$$K_y = K_1(z+h)^{\alpha}. \tag{1.15}$$

(In Davies's solution $h = 0$.) This relation had apparently two advantages. First, on physical grounds, it seemed at first sight, due to some degree of anisotropy in the turbulence very near the ground, to be reasonable to suppose that $K_y > K_z$ near the ground (that this idea leads to a wrong 'decay' index is clearly tied up with the concept given above, that the decay is governed by the spreading out at consecutively higher levels, as x increases, where the turbulence becomes more and more isotropic). Secondly, it enabled the two variables y and z in the diffusion equation (1.2) to be separated in the mathematical sense in a way similar to the present solution for $\alpha = \frac{1}{2}$. Sutton (1953, p. 283) has suggested that in fact a more realistic rate of decay will be obtained if K_y is taken equal to K_z (as is stated in (1.5)). As we have seen this is borne out by the solutions obtained in the subsequent sections.

Supported by the comparative failure of the above two solutions, different approaches to the problem have been developed. Notable amongst these are those due to Sutton (1934) and Davies (1954). Sutton adopted a statistical approach based on Taylor's theorem for the standard deviation $\sigma(T)$ of the distance travelled in a time T by an infinite succession of particles in a field of homogeneous turbulence. An expression for the concentration as a function of α was formulated in which the lateral dispersion was derived using Taylor's theorem; but except for this dependence on α and the consequential necessity of making the solution satisfy the continuity of matter equation, the expression is based on the solution (1.14) for constant velocity and K with height. In the case of a point-source at ground level, the expression is

$$C(x,y,z) = \frac{2Q}{\pi A_y\, A_z\, \bar{u} x^{2/(1+\alpha)}} \exp\left[-\left(\frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right], \tag{1.16}$$

where $\sigma_y^2 = \frac{1}{2} A_y\, x^{2/(1+\alpha)}, \qquad \sigma_z^2 = \frac{1}{2} A_z\, x^{2/(1+\alpha)}.$

It seems, by an investigation due to Knighting (see Sutton 1953) on 'turbulent diffusion as a random walk process', that the basic assumptions lead to a conception of the diffusive process not intrinsically assumed in Sutton's derivation. This must be due to the character of the assumptions themselves; for example, to basing the expression on the solution for constant velocity. To make but one comment on this assumption, the interpolation curves of § 7 indicate that in fact the concentration profiles are most sensitive to changes in α when α is small (near constant velocity). The result is that equation (1.16) represents a diffusion process which implies that the diffusion is being carried out mainly by those eddies that are smaller than, or of the same order of magnitude as, the 'spreads' σ_y and σ_z.

In the work of Davies (1954) this concept is openly assumed. The effect of the turbulence is represented by eddy diffusivity coefficients K_y, K_z;

for a source at ground level, the form of K_z remains as in equation (1.3), whereas K_y is made a function not only of z but also of y, namely

$$K_y = K_1 z^{\alpha} y^{1-2\alpha}, \tag{1.17}$$

so as to make the diffusion more rapid as the plume spreads laterally.

This form of K_y retains both the advantages of K_y in equation (1.15), and moreover the solutions of (1.2) have a more closely correct rate of decay for large x; the form is nevertheless open to considerable theoretical doubts. (Assumably, for an elevated source, K_z would likewise depend on the coordinates of the source.)

In considering this fundamental assumption, a clear distinction has to be made between two different kinds of problem, namely 'one-particle' problems and 'two-particle' problems. Thus our problem, in which we require the expectation concentration downwind of a source, that is the probability that a single particle leaving the source will arrive at the point in question, falls into the first category. On the other hand if we are interested in, say, the expectation of the maximum density in the plume at any time some fixed distance d downstream, regardless of where this maximum may be in the plane $x = d$, then we clearly have to know the way in which one particle is tending to diffuse away from its neighbour; this problem falls into the second category. Now one-particle problems of this kind are problems in which the particle is subjected to displacements by all eddies, large or small, whereas in two-particle problems those eddies whose characteristic size is greater than the separation of the particles are not of primary interest in the problem because they merely shift the two particles equally and do not contribute to the factor required, namely their separation. And so, if we consider the point-source plume at any particular time, those eddies that are smaller than the width of the plume are spreading the plume out relative to its own axis whereas the larger eddies are bodily shifting the plume from side to side and up and down, and are thereby introducing 'intermittancy' in the catch at any recording instrument, similar to that in the turbulence at the edge of a wake.

Thus in two-particle problems it is indeed necessary to choose eddy diffusivity coefficients dependent on the position of the source. But since the present problem is a one-particle problem in which all the eddies are everywhere contributing to the diffusion process, that is to the mean catch, it is necessary to choose coefficients independent of the source position, as in fact has been done in this paper.

2. Elevated line-source solution C_0

The infinitely long line-source lies across the wind in the horizontal y-direction (figure 1) and is elevated a height h above ground level. When z is the vertical coordinate measured with the source as origin the velocity and the eddy diffusivity coefficient are taken to satisfy equations (1.1) and

(1.4) respectively. The expectation concentration then satisfies the equation of diffusion

$$u_0(z+h)^\alpha \frac{\partial C}{\partial x} = K_0 \frac{\partial}{\partial z}\left[(h+z)^{1-\alpha}\frac{\partial C}{\partial z}\right] \tag{2.1}$$

with the following boundary conditions.

(i) In the plane of the source $x = 0$, C is zero except at the source, where it is infinite: in 'delta function' notation, $C = Q\,\delta(z)$.
(ii) The ground is impervious to smoke: on $z = -h, K_z \partial C/\partial z = 0$.
(iii) The concentration dies away at great heights: $C \to 0$ as $z \to \infty$.
(iv) The flux across any plane $x =$ constant is independent of the value of x, since no smoke is lost or created for $x > 0$:

$$\int_{-h}^{\infty} Cu\,dz = Q. \tag{2.2}$$

(The last condition is not independent of the others.)

For the sake of simplicity the mathematical manipulation is carried out with coordinates related to the real ones as follows:

$$z \to zh, \qquad x \to \frac{u_0}{K_0} h^{1+2\alpha} x, \qquad Q \to u_0 h^{1+\alpha} Q. \tag{2.3}$$

(the real coordinates are placed first). This results in elimination of the constants u_0, K_0 and h from (2.1) and (2.2).

The method of attack is to use Heaviside p-operators whereby the differential operator $\partial/\partial x$ is replaced by p; and $C(x,z)$ is replaced by $C(p,z)$, where

$$C(x,z) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} C(p,z)\frac{e^{px}}{p}\,dp. \tag{2.4}$$

Equation (2.1) becomes

$$\frac{\partial^2 C}{\partial z^2} + \frac{1-\alpha}{z+1}\frac{\partial C}{\partial z} - (z+1)^{2\alpha-1} pC = -(z+1)^{2\alpha-1} pQ\,\delta(z). \tag{2.5}$$

The right-hand side may be put equal to zero provided the equation is solved separately in the two regions $z > 0$ and $z < 0$. The equation is then Bessel's equation, the solutions of which, satisfying (2.2 i, ii, iii), are

$$\left.\begin{aligned} C(p,z) &= A(p) r^{\alpha/(1+2\alpha)} K_{\alpha/(1+2\alpha)}(r) \qquad (z > 0),\\ C(p,z) &= D(p) r^{\alpha/(1+2\alpha)} I_{-\alpha/(1+2\alpha)}(r) \qquad (-1 < z < 0),\end{aligned}\right\} \tag{2.6}$$

where
$$r = \frac{2}{1+2\alpha}\sqrt{p}(z+1)^{(1+2\alpha)/2} \tag{2.7}$$

(throughout this paper $K_\nu(z)$ is defined as $\frac{1}{2}\pi \operatorname{cosec} \nu\pi\{I_{-\nu}(z) - I_\nu(z)\}$). Also, the two solutions must be equal on $z = 0$; the gradients do not, however, have to be continuous. Thus if $r = q$ on $z = 0$ we have

$$K_{\alpha/(1+2\alpha)}(q)\,A(p) = I_{-\alpha/(1+2\alpha)}(q)\,D(p). \tag{2.8}$$

Finally, applying (2.2 iv) and using the recurrence relation (Watson 1944, p. 80, eqn. (20)),

$$D(p) = Q\sqrt{p}\left(\frac{2\sqrt{p}}{1+2\alpha}\right)^{1/(1+2\alpha)} q^{\alpha/(1+2\alpha)} K_{\alpha/(1+2\alpha)}(q); \tag{2.9}$$

and similarly for $A(p)$. The operational forms (2.6) have to be interpreted to recover $C(x, z)$ by application of the integral (2.4). Clearly the two forms are very similar and, in fact, they have identical interpretations. Thus, considering only the solution for $z > 0$,

$$C(x,z) = \frac{2}{1+2\alpha} Q(z+1)^{\alpha/2} \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{px} I_{-\alpha/(1+2\alpha)}\left(\frac{2}{1+2\alpha}\sqrt{p}\right) \times$$
$$\times K_{\alpha/(1+2\alpha)}\left(\frac{2}{1+2\alpha} a\sqrt{p}\right) dp \quad (2.10)$$

where $a = (z+1)^{(1+2\alpha)/2}$. This may be evaluated by deforming the contour to one whose significant part is a loop round the negative real axis (figure 6). Since the integrand has zero residue at $p = 0$ and gives no contribution

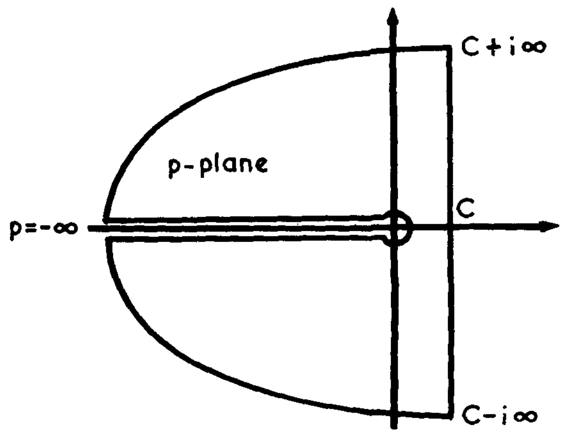

Figure 6. The contour from $C-i\infty$ to $C+i\infty$ is deformed into the semicircle at infinity and a loop round the negative real axis and the origin.

on the circle at infinity, (2.10) may be represented as the sum of two integrals along the negative real axis, which may be combined to give

$$C(x,z) = \frac{1+2\alpha}{2} Q(z+1)^{\alpha/2} \int_0^\infty q J_{-\alpha/(1+2\alpha)}(q) J_{-\alpha/(1+2\alpha)}(aq) \times$$
$$\times \exp\{-\tfrac{1}{4}(1+2\alpha)^2 x q^2\}\, dq. \quad (2.11)$$

This integral is a special case of Weber's second exponential integral (Watson 1944, p. 395)

$$\int_0^\infty \exp(-p^2 t^2) J_\nu(at) J_\nu(bt) t\, dt = \frac{1}{2p^2} \exp\left(-\frac{a^2+b^2}{4p^2}\right) I_\nu\left(\frac{ab}{2p^2}\right), \quad (2.12)$$

which, when applied to (2.11), and reverting to the real variables (2.3), yields the final solution

$$C_0 = \frac{Q}{(1+2\alpha)} \frac{(zh+h^2)^{\alpha/2}}{K_0 x} \exp\left[-\frac{u_0(z+h)^{1+2\alpha} + u_0 h^{1+2\alpha}}{K_0(2\alpha+1)^2 x}\right] \times$$
$$\times I_{-\alpha/(1+2\alpha)}\left[\frac{2u_0(zh+h^2)^{(1+2\alpha)/2}}{K_0(2\alpha+1)^2 x}\right]. \quad (2.13)$$

The general properties of this solution have been discussed in the introduction.

3. Elevated point source $\alpha = \frac{1}{2}$

This is the single exact solution for the complete point-source problem. The particular virtue about $\alpha = \frac{1}{2}$, $\mu = 0$ is that u and K have the same functional form. This enables the variables y and z to be separated in the diffusion equation.

The equation of diffusion is (1.2) with u, K_z and K_y given by (1.1), (1.4) and (1.5), respectively, with boundary conditions identical with (2.2) except for (iv) which is changed to

$$\int_{-h}^{\infty}\int_{-\infty}^{\infty} Cu\, dydz = Q. \tag{3.1}$$

As in §2 it is found convenient to use related coordinates:

$$x \to \frac{u_0}{K_0} h^{1+2\alpha} x, \quad z \to hz, \quad y \to \sqrt{\left(\frac{K_1}{K_0}\right)} hy, \quad Q \to u_0 h^{2+\alpha} \sqrt{\left(\frac{K_1}{K_0}\right)} Q \tag{3.2}$$

(the real coordinates are placed first), where K_1, as used in (1.6), is equal to K_0 when $\mu = 0$. The equation of diffusion is thus

$$(z+1)^{1/2}\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left[(z+1)^{1/2}\frac{\partial C}{\partial z}\right] + (z+1)^{1/2}\frac{\partial^2 C}{\partial y^2}. \tag{3.3}$$

A solution of the form

$$C = X(x,y)\, Y(\eta), \qquad \eta = \frac{y^2}{4x}, \tag{3.4}$$

is sought, whereby (3.3) can be separated into two distinct equations linked only by the parameter m:

$$\frac{\partial^2 X}{\partial z^2} + \frac{1}{2(z+1)}\frac{\partial X}{\partial z} - \frac{\partial X}{\partial x} + \frac{m}{x} X = 0, \tag{3.5}$$

$$\frac{\partial^2 Y}{\partial \eta^2} + \left(1 + \frac{1}{2\eta}\right)\frac{\partial Y}{\partial \eta} - \frac{m}{\eta} Y = 0. \tag{3.6}$$

The solution satisfying the boundary conditions is found by putting $m = 0$. Then (3.6) can be solved to give

$$\frac{\partial Y}{\partial \eta} = \frac{1}{\sqrt{\eta}} e^{-\eta},$$

and since by inspection of (3.3) we see that $\partial C/\partial y$ is also a solution then so is

$$X(x,z)\frac{1}{\sqrt{x}} e^{-y^2/4x}, \tag{3.7}$$

where $X(x,z)$ satisfies (3.5) with $m = 0$. This is exactly the same differential equation as (2.1) with $\alpha = \frac{1}{2}$; also $X(x,z)$ must satisfy the same boundary conditions (2.2) except that the flux Q must be altered to $Q/2\sqrt{\pi}$. Thus the solution, reverting to the real variables (3.2), is

$$C(x,z) = \frac{Q}{4\sqrt{\pi}}\frac{(zh+h^2)^{1/4}}{(K_0 x)^{3/2}} \exp\left[-\frac{y^2+h^2+(z+h)^2}{4K_0 x/u_0}\right] I_{-1/4}\left(\frac{zh+h^2}{2K_0 x/u_0}\right). \tag{3.8}$$

In the generalized case when K_y satisfies (1.6), the exact solution is found in an exactly similar manner for the case

$$\alpha = \tfrac{1}{2}(1+\mu). \tag{3.9}$$

The solution is then

$$C = \frac{Q}{2(1+2\alpha)}\frac{1}{\sqrt{(\pi K_1)}}\frac{(zh+h^2)^{\alpha/2}}{K_0 x^{3/2}} \times$$
$$\times \exp\left[-\frac{\frac{1}{4}(1+2\alpha)^2(K_0/K_1)y^2+(z+h)^{1+2\alpha}+h^{1+2\alpha}}{(1+2\alpha)^2 K_0 x/u_0}\right]\times$$
$$\times I_{-\alpha/(1+2\alpha)}\left(\frac{2u_0(zh+h^2)^{(1+2\alpha)/2}}{(1+2\alpha)^2 K_0 x}\right). \tag{3.10}$$

The important thing to note in this section is the Gaussian form of Y, as this is the basis of the subsequent sections.

4. Point-source at ground level

With the variables changed, for the sake of simplicity, as follows,

$$x \to \frac{u_0}{K_0}x, \qquad z \to z, \qquad y \to \sqrt{\left(\frac{K_1}{K_0}\right)}y, \qquad Q \to u_0\sqrt{\left(\frac{K_1}{K_0}\right)}Q, \tag{4.1}$$

the equation of diffusion (1.2) is

$$z^{\alpha}\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left(z^{1-\alpha}\frac{\partial C}{\partial z}\right)+z^{1-\alpha+\mu}\frac{\partial^2 C}{\partial y^2}. \tag{4.2}$$

The approach of this section has been discussed in the Introduction and involves determining exact solutions for C_0 (see (1.7)) and C_2 (see (1.8)) which may then be utilized to find the concentration C by the formulae (1.9) and (1.10).

C_0, being the solution of the line-source problem, is already determined (§ 2), and in the special case of zero source-elevation h is

$$C_0 = \Lambda x^{-(1+\alpha)/(1+2\alpha)}\exp\left[-\frac{z^{1+2\alpha}}{(1+2\alpha)^2 x}\right], \tag{4.3}$$

where

$$\Lambda = \frac{Q}{(1+2\alpha)^{1/(1+2\alpha)}[-\alpha/(1+2\alpha)]!}. \tag{4.4}$$

The second function, C_2, satisfies the equation

$$z^{\alpha}\frac{\partial C_2}{\partial x} = \frac{\partial}{\partial z}\left[z^{1-\alpha}\frac{\partial C_2}{\partial z}\right]+2\Lambda z^{1-\alpha+\mu}x^{-(1+\alpha)/(1+2\alpha)}\exp\left[-\frac{z^{1+2\alpha}}{(1+2\alpha)^2 x}\right]. \tag{4.5}$$

Writing

$$\eta = \frac{z^{1+2\alpha}}{(1+2\alpha)^2 x}, \qquad a = \frac{1+\alpha}{1+2\alpha}, \qquad b = \frac{2+\mu}{1+2\alpha}, \tag{4.6}$$

then the solution of (4.5) is obtained by expressing C_2 in the form

$$C_2 = 2\Lambda(1+2\alpha)^{2b-2}x^{b-a}e^{-\eta}G(\eta), \tag{4.7}$$

where $G(\eta)$ satisfies the ordinary differential equation

$$\frac{d^2G}{d\eta^2}+\left(\frac{a}{\eta}-1\right)\frac{dG}{d\eta}-\frac{b}{\eta}G = -\eta^{b-2}. \tag{4.8}$$

The homogeneous equation is the equation for the confluent hypergeometric function and has the two independent solutions (excluding $\alpha = 0$)

$$ {}_1F_1(b;a;\eta), $$

$$ U(b;a;\eta) = \frac{(-a)!}{(b-a)!}\,{}_1F_1(b;a;\eta) + \frac{(a-1)!}{(b-1)!}\eta^{1-a}\,{}_1F_1(1-a+b;2-a;\eta). \quad (4.9) $$

We choose $U(b;a;\eta)$ for the second solution since it is $o(e^{\eta})$ at $\eta = \infty$ (Jeffreys & Jeffreys 1946, p. 579). The particular integral of (4.8) can be expressed, by the usual formula given in the theory of 'variation of parameters', in terms of the right-hand side and the Wronskian

$$ W = -\frac{(a-1)!}{(b-1)!}\eta^{-a}e^{\eta}, \quad (4.10) $$

so that the full solution is

$$ C_2 = 2\Lambda(1+2\alpha)^{2b-2}x^{b-a}e^{-\eta}\frac{(b-1)!}{(a-1)!} \times $$
$$ \times \{U(b;a;\eta)[I_1+A] - {}_1F_1(b;a;\eta)[I_2+B]\}, \quad (4.11) $$

where I_1 and I_2 are indefinite integrals arising from the particular integral,

$$ \left.\begin{aligned} I_1 &= \int_0^{\eta} \eta^{a+b-2}e^{-\eta}\,{}_1F_1(b;a;\eta)\,d\eta, \\ I_2 &= \int_0^{\eta} \eta^{a+b-2}e^{-\eta}U(b;a;\eta)\,d\eta, \end{aligned}\right\} \quad (4.12) $$

and A and B are constants to be determined.

The boundary conditions that (4.11) must satisfy are (2.2i, ii, iii).

Thus $C_2 \to 0$ as $\eta \to \infty$, which in terms of $G(\eta)$ implies that $G = o(e^{\eta})$ for large η. In this limit, the second term is dominant and would be $O(e^{\eta})$ unless $B = -I_2(\infty)$. $I_2(\infty)$ may be evaluated by use of the integral representation of $U(b;a;\eta)$, namely

$$ U(b;a;\eta) = \frac{1}{(b-a)!}\int_0^{\infty} e^{-v}v^{b-a}(v+\eta)^{-b}\,dv, \quad (4.13) $$

to give

$$ I_2(\infty) = -B = \frac{(b-1)!\,(a+b-2)!}{(2b-1)!}. \quad (4.14) $$

Applying the zero-flux condition on $z = 0$, $\eta = 0$, which is

$$ \eta^{a}\frac{\partial C}{\partial \eta} \quad \text{on} \quad \eta = 0, \quad (4.15) $$

to (4.11), the only non-zero contribution is the term containing the factor

$$ \eta^{a}A\frac{\partial}{\partial \eta}U(b;a;\eta). $$

And so, to satisfy (4.15), we have

$$ A \equiv 0, \quad (4.16) $$

Hence

$$G = \frac{(b-1)!}{(a-1)!}\Big[U(b;a;\eta)\int_0^\eta \eta^{a+b-2}e^{-\eta}\,{}_1F_1(b;a;\eta)\,d\eta + $$
$$+{}_1F_1(b;a;\eta)\left\{\frac{(b-1)!\,(a+b-2)!}{(2b-1)!} - \int_0^\eta \eta^{a+b-2}e^{-\eta}\,U(b;a;\eta)\,d\eta\right\}\Big]. \quad (4.17)$$

This is further simplified by expanding the integrands in powers of η, integrating term by term, and then collecting like powers of η to give the rapidly convergent series

$$G(\eta) = \frac{(b-1)!\,(b+a-2)!}{(2b-1)!}\left[\frac{(b-1)!}{(a-1)!}\,{}_1F_1(b;a;\eta) - \eta^b V(b;a;\eta)\right], \quad (4.18)$$

where

$$V(b;a;\eta) = \sum_{r=0}^{\infty}\frac{(2b+r-1)!}{(b+r)!\,(b+a+r-1)!}\eta^r. \quad (4.19)$$

The evaluation of this solution is sufficiently simple to warrant its use. Generally no more than eight terms are required to give the solution correct to four places.

In terms of the real variables (4.1), the 'spread' function is

$$C_2 = 2Q\sqrt{\left(\frac{K_1}{K_0}\right)}\frac{K_0^{b-a}}{u_0^{b-a+1}}(1+2\alpha)^{(3b-4)/2}\frac{(b-1)!\,(b+a-2)!}{(a-1)!\,(2b-1)!}x^{b-a}\times$$
$$\times e^{-\eta}\left[\frac{(b-1)!}{(a-1)!}\,{}_1F_1(b;a;\eta) - \eta^b V(b;a;\eta)\right], \quad (4.20)$$

with

$$\eta = \frac{u_0 z^{1+2\alpha}}{(1+2\alpha)^2 K_0 x}, \qquad a = \frac{1+\alpha}{1+2\alpha}, \qquad b = \frac{2+\mu}{1+2\alpha}. \quad (4.21)$$

Four special cases are known in which the solutions of (4.5) can be obtained very much more directly. These solutions are obtained in terms of simple functions and can be shown to be included in the general form (4.20). The four cases correspond to values of $\alpha = 0$, $\frac{1}{3}$, $\frac{1}{2}$ and 1, with $\mu = 0$:

$$\alpha = 0, \qquad C_2 = \frac{QK_0}{3u_0^2}xe^{-\eta}(1+2\eta), \qquad \eta = \frac{u_0 z}{K_0 x}; \quad (4.22)$$

$$\left.\begin{aligned}\alpha = \tfrac{1}{3}, \qquad & C_2 = \frac{2}{7}\frac{Q}{K^{2/5}u_0^{3/5}}\left(\frac{3}{5}\right)^{1/5}\frac{x^{2/5}}{(-\frac{1}{5})!}\times \\ & \times e^{-\eta}\left[5\eta^{1/5} + \frac{(\frac{1}{5})!}{(-\frac{1}{5})!}U(\tfrac{6}{5};\tfrac{4}{5};\eta)\right], \qquad \eta = \frac{9u_0 z^{5/3}}{25K_0 x};\end{aligned}\right\} \quad (4.23)$$

$$\alpha = \tfrac{1}{2}, \qquad C_2 = 2\frac{K_0}{u_0}xC_0 \quad (4.24)$$

(this being the exact solution of § 3);

$$\alpha = 1, \qquad C_2 = \frac{2Q}{u_0}I^*\left(\frac{u_0 z^3}{9K_0 x}, -\frac{1}{3}\right), \quad (4.25)$$

$$I^*(\eta, \xi) = 1 - \int_0^\eta \eta^\xi e^{-\eta}\,d\eta \Big/ \int_0^\infty \eta^\xi e^{-\eta}\,d\eta \quad (4.26)$$

(Pearson 1922).

We noted in the Introduction that the rate of decrease of concentration as x increases for large x is considered to be a valid test of any proposed expression. Equations (1.10), (4.3) and (4.20) show that

$$C \sim x^{-(b+2a)/2} = x^{-(2+\alpha+\frac{1}{2}\mu)/(1+2\alpha)}, \tag{4.27}$$

so when $\mu = 0$ and $\alpha = \frac{1}{7}$ (the expected value in neutral stability)

$$C \sim x^{-5/3}, \tag{4.28}$$

which agrees with those values which have been obtained by experiment (Sutton 1947).

5. Reciprocal theorem

The reciprocal theorem, which is of vital importance in extending the work to the problem of the elevated source, may be stated as follows.

> The concentration at $\mathbf{x}'$ due to a source at $\mathbf{x}''$, with the flow in the positive x_1-direction, is equal to the concentration at $\mathbf{x}''$ due to an identical source at $\mathbf{x}'$ when the direction of the flow is reversed.

The proof of the theorem runs thus.

Consider the partial differential equation

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial G}{\partial x_j}\right) + B_i\frac{\partial G}{\partial x_i} = 0 \tag{5.1}$$

(with summation over repeated suffixes), where G is a Green's function with centre $\mathbf{x}''$ and which satisfies the boundary conditions

$$\left.\begin{aligned} B_i n_i &= 0 \\ K_{ij}\frac{\partial G}{\partial x_j}\, n_i &= 0 \end{aligned}\right\} \text{on the ground} \tag{5.2}$$

(n_i are the direction cosines of the normal to the bounding surface—for the horizontal ground $n_i = (0, 0, 1)$). Let this equation be represented by $LG = 0$ where L is an operator. Also let the adjoint equation, with the same boundary conditions,

$$\frac{\partial}{\partial x_i}\left(K_{ji}\frac{\partial G_a}{\partial x_j}\right) - \frac{\partial}{\partial x_i}(B_i G_a) = 0, \tag{5.3}$$

be represented by $MG_a = 0$. G_a is a Green's function of the adjoint equation with centre $\mathbf{x}'$. Then

$$\begin{aligned} 0 &= \int_V [G_a LG - GMG_a]\, d\mathbf{x} \\ &= \int_V \left[G_a\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial G}{\partial x_j}\right) - G\frac{\partial}{\partial x_i}\left(K_{ji}\frac{\partial G_a}{\partial x_j}\right) + \frac{\partial}{\partial x_i}(GB_iG_a)\right] d\mathbf{x} \\ &= \int_S \left[G_a K_{ij}\frac{\partial G}{\partial x_j}n_i - GK_{ji}\frac{\partial G_a}{\partial x_j}n_i\right] dS + \int_S GG_aB_in_i\, dS, \end{aligned} \tag{5.4}$$

by the application of Green's theorem and the divergence theorem. Let the bounding surface S be the ground, the hemisphere at infinity and the two small spheres excluding the singularities $\mathbf{x}'$ and $\mathbf{x}''$. Since $B_i n_i = 0$

on the ground and $G \sim 1/r$ (r = distance from source $\mathbf{x}''$) the second integral makes no contribution.

The first integral makes a contribution only on the small spheres: the first term on the one around $\mathbf{x}''$, the second on the one around $\mathbf{x}'$. Provided these spheres are sufficiently small, G_a is constant on the one at $\mathbf{x}''$ and G on the one at $\mathbf{x}'$. Since the flux from the two sources are equal, the right-hand side of (5.4) reduces to

$$Q[G_a(\mathbf{x}'', \mathbf{x}') - G(\mathbf{x}', \mathbf{x}'')], \tag{5.5}$$

where Q is this flux. Thus

$$G(\mathbf{x}', \mathbf{x}'') = G_a(\mathbf{x}'', \mathbf{x}'); \tag{5.6}$$

in words, the Green's functions of the equation and its adjoint are the same if the variables are interchanged.

Now if

$$K_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_y(z) & 0 \\ 0 & 0 & K_z(z) \end{bmatrix} \quad \text{and} \quad B_i = (u(z), 0, 0), \tag{5.7}$$

then $LG = 0$ is the diffusion equation with flow in the positive direction of x and $MG_a = 0$ is the diffusion equation with flow in the negative direction of x. Also, $G(\mathbf{x}', \mathbf{x}'')$ represents the concentration at $\mathbf{x}'$ due to a source at $\mathbf{x}''$ with flow in the positive direction, and $G_a(\mathbf{x}'', \mathbf{x}')$ represents the concentration at $\mathbf{x}''$ due to a source at $\mathbf{x}'$ with flow in the reversed direction. We have $G(\mathbf{x}', \mathbf{x}'') = G_a(\mathbf{x}'', \mathbf{x}')$ which is the theorem to be proved.

This theorem, as discussed in the Introduction, enables the ground level concentrations to be determined downwind of an elevated point-source from the concentrations at arbitrary level downwind of a point-source on the ground. In particular it is used to determine C_2 at ground level (everything above applies to C_0 and C_2 as well as to C). For source-elevation h this is obtained from (4.20) and (4.21) by putting $z = h$.

6. Elevated point-source: 'spread' function C_2

It would be desirable to find C_2 for all heights (although of course it is the ground level value in which we are particularly interested). The complete solution has already been found in § 3 for $\alpha = \frac{1}{2}$. This section, following an approach similar to that used in § 2 to find C_0, yields the complete solution for $\alpha = 0$. It seems possible that the cases $\alpha = \frac{1}{3}$ and $\alpha = 1$ could also be treated in this way.

With the transformation of variables (3.2) together with changing C_2 to $h^3 C_2$, the modified C_2 satisfies the equation

$$(z+1)^{\alpha} \frac{\partial}{\partial x} C_2 = \frac{\partial}{\partial z}\left[(1+z)^{1-\alpha} \frac{\partial}{\partial z} C_2\right] + 2(1+z)^{1-\alpha} C_0, \tag{6.1}$$

when μ is zero. In p-operational form (see § 2), this becomes

$$\frac{\partial^2 C_2}{\partial z^2} + \frac{1-\alpha}{1+z} \frac{\partial C_2}{\partial z} - (z+1)^{2\alpha-1} p C_2 = -2C_0, \tag{6.2}$$

since $C_2 = 0$ on $x = 0$. As before, put

$$r = \frac{2}{1+2\alpha} \sqrt{p}(z+1)^{(1+2\alpha)/2}, \tag{6.3}$$

and let

$$\left.\begin{aligned} A(p) &= 2Q\left(\frac{1+2\alpha}{2}\right)^{(1-5\alpha)/(1+2\alpha)} p^{(3\alpha-2)/(2+4\alpha)} I_{-\alpha/(1+2\alpha)}\left[\frac{2}{1+2\alpha}\sqrt{p}\right], \\ A'(p) &= 2Q\left(\frac{1+2\alpha}{2}\right)^{(1-5\alpha)/(1+2\alpha)} p^{(3\alpha-2)/(2+4\alpha)} K_{\alpha/(1+2\alpha)}\left[\frac{2}{1+2\alpha}\sqrt{p}\right]. \end{aligned}\right\} \tag{6.4}$$

Then (6.2) becomes

$$\left.\begin{aligned} \frac{\partial^2 C_2}{\partial r^2} + \frac{1}{1+2\alpha}\frac{1}{r}\frac{\partial C_2}{\partial r} - C_2 &= -A(p) r^{(2-3\alpha)/(1+2\alpha)} K_{\alpha/(1+2\alpha)}(r) \quad (z>0), \\ &= -A'(p) r^{(2-3\alpha)/(1+2\alpha)} I_{-\alpha/(1+2\alpha)}(r) \quad (z<0), \end{aligned}\right\} \tag{6.5}$$

by the application of (2.6), (2.8) and (2.9). The complementary functions are the same as in §2. The trouble arises in attempting to evaluate the particular integral. For $z > 0$, say, the particular integral is

$$\text{P.I.} = A(p) r^{\alpha/(1+2\alpha)} K_{\alpha/(1+2\alpha)}(r) \int_0^r s^{(3-2\alpha)/(1+2\alpha)} K_{\alpha/(1+2\alpha)}(s) I_{\alpha/(1+2\alpha)}(s)\, ds - \\ - A(p) r^{\alpha/(1+2\alpha)} I_{\alpha/(1+2\alpha)}(r) \int_0^r s^{(3-2\alpha)/(1+2\alpha)} K^2_{\alpha/(1+2\alpha)}(s)\, ds. \tag{6.6}$$

These integrals may be evaluated in terms of simple functions only for the four values of α: $\alpha = 0, \frac{1}{3}, \frac{1}{2}$ and 1 (cf. §4). The solution for $\alpha = \frac{1}{2}$ has been carried through and is in complete agreement with §3. The two cases of $\alpha = \frac{1}{3}$ and $\alpha = 1$ have been explored; however in taking the interpretations of the solutions obtained, it was found necessary to put $z = -1$ (corresponding to ground level). The results were in full agreement with the solutions obtained by application of the reciprocal theorem to (4.23) and (4.26).

For $\alpha = 0$, the analysis can be carried through without any restriction. Since $\alpha = 0$ is of particular interest, being close to $\alpha = \frac{1}{7}$, and also because it is typical (although quite different in detail) of the other three solutions, the solution will be given below.

When $\alpha = 0$, equations (6.4) reduce to

$$\left.\begin{aligned} A(p) &= \frac{Q}{p} I_0(2\sqrt{p}) \\ A'(p) &= \frac{Q}{p} K_0(2\sqrt{p}) \end{aligned}\right\} \tag{6.7}$$

and (6.6) reduces to

$$\text{P.I.} = AK_0(r) \int^r s^3 I_0 K_0\, ds - AI_0(r) \int^r s^3 K_0^2\, ds. \tag{6.8}$$

A similar expression is obtained for $z < 0$. If C_n and $\mathscr{C}_n$ represent modified Bessel functions of purely imaginary argument (I_n and $K_n \cos n\pi$) then

$$\int^r s^3 C_0(s)\mathscr{C}_0(s)\, ds = \tfrac{1}{12} r^4 [3C_0(r)\mathscr{C}_0(r) - 2C_1(r)\mathscr{C}_1(r) - C_2(r)\mathscr{C}_2(r)] \tag{6.9}$$

E 2

(Watson 1944, p. 136). This relation is used to express the particular integrals (6.8) in terms of products of three Bessel functions. By re-arranging the terms, the Wronskian relationship (Watson 1944, p. 80, eqn. (20)

$$K_\nu(r)I_{\nu+1}(r)+I_\nu(r)K_{\nu+1}(r) = 1/r \tag{6.10}$$

may be used to give

$$\left.\begin{aligned}\text{P.I.} &= \frac{A}{6}r^3\left[K_1(r)+\frac{K_2(r)}{r}\right] \qquad (z>0),\\ &= -\frac{A'}{6}r^3\left[I_1(r)-\frac{I_2(r)}{r}\right] \qquad (z<0).\end{aligned}\right\} \tag{6.11}$$

Thus the solution which has the correct behaviour at $z = \infty$ is

$$\left.\begin{aligned}C_2 &= L(p)K_0(r)+\frac{A}{6}r^3\left(K_1+\frac{K_2}{r}\right) \qquad (z>0),\\ C_2 &= F(p)K_0(r)-G(p)I_0(r)-\frac{A'}{6}r^3\left(I_1-\frac{I_2}{r}\right) \qquad (z<0).\end{aligned}\right\} \tag{6.12}$$

Where $L(p)$, $F(p)$ and $G(p)$ have to be determined from the conditions:

(i) no flux across $z = -1$, $r = 0$, i.e. $\partial C_2/\partial r = 0$, giving $F(p) = 0$;

(ii) on $z = 0$ the two solutions have to be equal, so that, if $r = q = 3\sqrt{p}$ on $z = 0$, we have

$$\left.\begin{aligned}C_2 = &-\tfrac{2}{3}Q\{qI_1(q)-I_2(q)\}K_0(r)+\\ &+\frac{2r^2}{3q^2}QI_0(q)\{rK_1(r)+K_2(r)\} \qquad (z>0),\\ C_2 = &\tfrac{2}{3}Q\{qK_1(q)+K_2(q)\}I_0(r)-\\ &-\frac{2r^2}{3q^2}QK_0(q)\{rI_1(r)-I_2(r)\} \qquad (z<0).\end{aligned}\right\} \tag{6.13}$$

This equation (6.13) has to be reduced to a form in which it may be interpreted; we are limited to reducing the two relations to sums of pairs of Bessel functions of equal order, or the derivatives of such pairs. The clue, as to what form to look for, is given by the term $(\ )r^3QI_0(q)K_1(r)$. It suggests that the form should contain derivatives up to and including the third. By use of the differential equations satisfied by the Bessel functions and also the recurrence relations such as

$$I_2(q) = I_0(q)-\frac{2}{q}I_1(q), \qquad I_1(q) = \frac{d}{dq}I_0(q), \tag{6.14}$$

the equation (6.13) may be simplified to

$$\left.\begin{aligned}C_2 &= -\tfrac{2}{3}Q\left[q\frac{d^3}{dq^3}\{I_0(q)K_0(r)\}+3r\frac{d}{dq}\{I_1(q)K_1(r)\}\right] \qquad (z>0),\\ C_2 &= -\tfrac{2}{3}Q\left[q\frac{d^3}{dq^3}\{I_0(r)K_0(q)\}+3r\frac{d}{dq}\{I_1(r)K_1(q)\}\right] \qquad (z<0).\end{aligned}\right\} \tag{6.15}$$

The interpretation will be carried out for $z < 0$. The method is exactly similar for $z > 0$ and it can be seen that the interpretations would finally be identical, as they were in §2.

The second term has the interpretation

$$\mathscr{I}_2 = -Q(z+1)^{1/2}\frac{2}{\pi i}\int_{C-i\infty}^{C+i\infty}\frac{\partial}{\partial p}\{I_1(r)K_1(q)\}e^{px}\,dp, \tag{6.16}$$

and on integrating by parts this is

$$\mathscr{I}_2 = 2Q(z+1)^{1/2}\frac{x}{\pi i}\int_{C-i\infty}^{C+i\infty} I_1(r)K_1(q)e^{px}\,dp. \tag{6.17}$$

Following the same process as in §2, we find

$$\mathscr{I}_2 = 2Q(z+1)^{1/2}\exp\left[-\frac{z+2}{x}\right]I_1\left(\frac{2\sqrt{(z+1)}}{x}\right). \tag{6.18}$$

The first term in (6.15) ($z < 0$) has the interpretation

$$\mathscr{I}_1 = -\tfrac{2}{3}Q\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} q\frac{d^3}{dq^3}\{I_0(r)K_0(q)\}\frac{e^{px}}{p}\,dp, \tag{6.19}$$

which again may be integrated by parts to give

$$\mathscr{I}_1 = \tfrac{2}{3}Q\frac{x}{\pi i}\int_{C-i\infty}^{C+i\infty}(x^2p+\tfrac{3}{2}x)I_0(r)K_0(q)e^{px}\,dp \tag{6.20}$$

$$= \left(\tfrac{4}{3}Qx^3\frac{d}{dx}+2Qx^2\right)\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} I_0(r)K_0(q)e^{px}\,dp. \tag{6.21}$$

Again following the same process as in §2, we finally obtain

$$C_2 = \tfrac{2}{3}Q\left\{\left[z+2+\frac{x}{2}\right]I_0\left(\frac{2\sqrt{(z+1)}}{x}\right)+\right.$$
$$\left.+\sqrt{(z+1)}I_1\left(\frac{2\sqrt{(z+1)}}{x}\right)\right\}\exp\left[-\frac{z+2}{x}\right] \tag{6.22}$$

for all z. This solution reduces to (4.22) with $\eta = u_0 h/K_0 x$ when the real variables are substituted back into (6.22):

$$C_2 = \frac{QK_0}{3u_0^2}xe^{-\xi}\{(1+2\xi)I_0(\rho)+\rho I_1(\rho)\} \tag{6.23}$$

where $$\xi = \frac{z+2h}{K_0 x/u_0},\qquad \rho = \frac{2u_0\sqrt{(zh+h^2)}}{K_0 x}$$

and when z is put equal to its value at the ground $z = -h$. Equation (6.23) is used together with (2.13) in (1.9) and (1.10) to plot the concentration at three different levels (figure 10) in the next section.

7. Results

The results of the previous sections are perhaps best presented in graphical form, when the differences between the curves are most readily appreciated. Since the graphs are more or less self-explanatory it will suffice to tabulate the main points in a very concise way.

The profile for the elevated line-source, when $\alpha = \frac{1}{7}$, has been given in figure 2 in §1.

For the point-source problem, the reciprocal theorem has been applied to equation (4.20) to give the ground level concentration when $\alpha = \frac{1}{7}$. This profile and the four simpler cases ((4.22) to (4.26)) are given on the same graph (figure 7), and are sufficient to deduce, for general α, interpolation curves for the main features, namely, the magnitude of the maximum

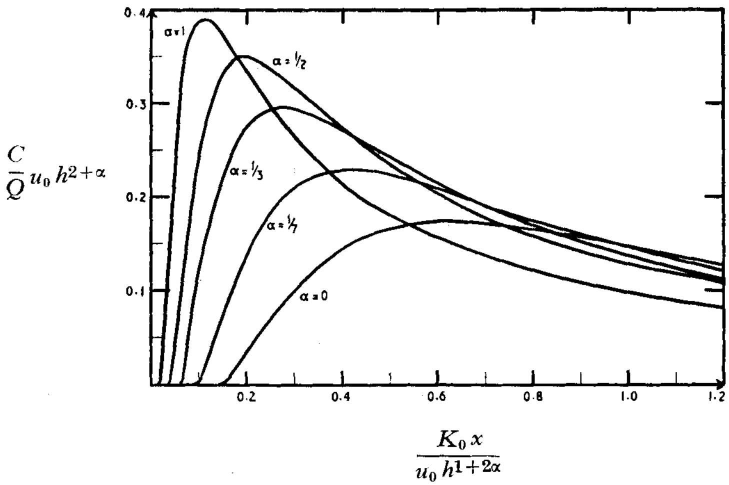

Figure 7. The concentration at ground level along the line $y = 0$, as a function of x, the distance downstream, due to an elevated point source. The five curves correspond to values of $\alpha = 0, \frac{1}{7}, \frac{1}{3}, \frac{1}{2}$ and 1.

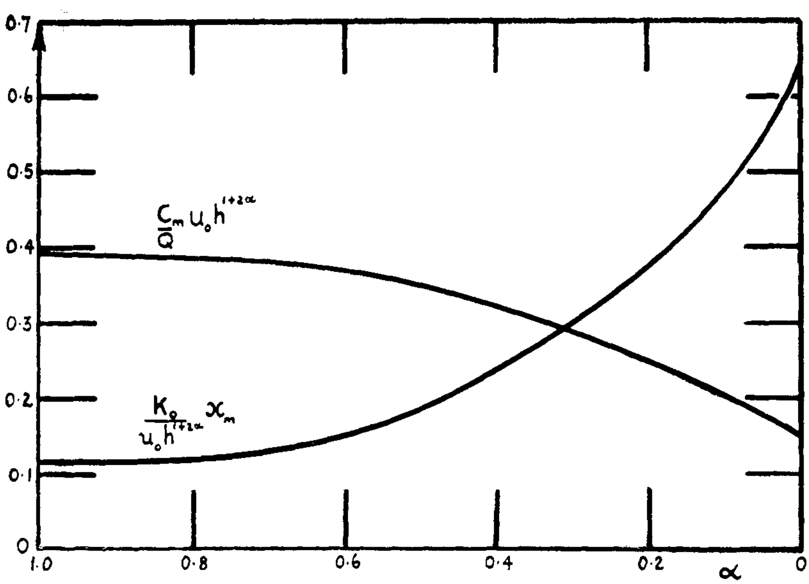

Figure 8. Interpolation curves showing the maximum concentration C_m at ground level and the downstream distance x_m from the elevated source to the position of that maximum, plotted as functions of the parameter α.

concentration and its distance downstream (figure 8). These curves indicate that the profiles are most sensitive to changes in α when α is small. The combined effect due to the variation in the shear and in K as α increases can be summarized as follows:

(i) the maximum goes up;
(ii) the maximum occurs earlier;
(iii) the maximum is 'peakier';
(iv) the plume strikes the ground earlier;
(v) the concentration falls off for large x more slowly.

Figure 9 shows the variation of concentration for $\alpha = 0$ at three different levels: at ground level, at source height, and at twice the height of the

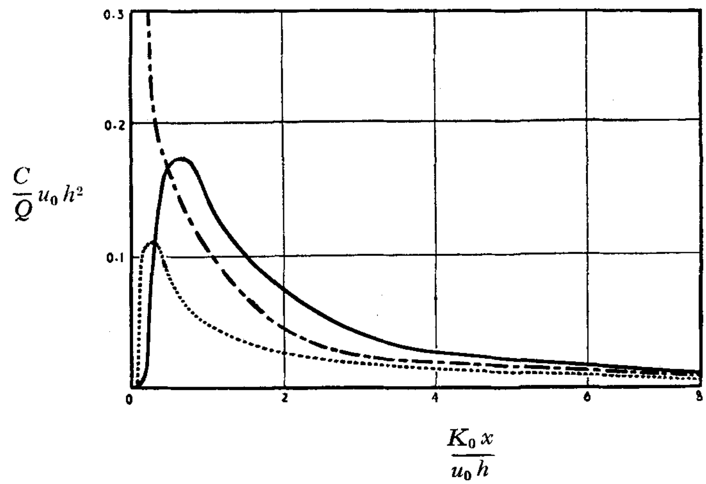

Figure 9. The concentration on $y = 0$, downwind of an elevated point source when $\alpha = 0$. It is plotted at three heights: at the ground ——, at source height —·— and at twice the height of the source ······ .

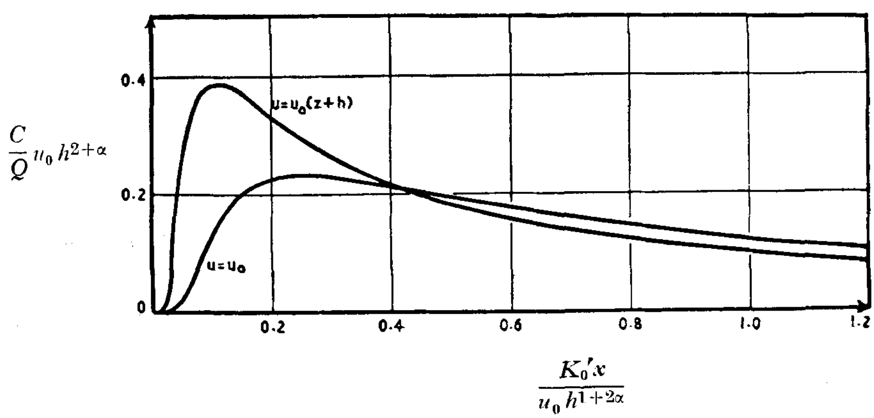

Figure 10. Both curves give the concentration when the eddy diffusivity coefficient is constant with height. The 'peakier' of the two curves applies when the velocity varies linearly with height, the other when the velocity is constant.

source, $z = h$. Initially, as we saw in the Introduction (figure 3), the concentration is greater at $z = h$ than at $z = -h$. But later, due to the reflective nature of the ground and to the greater lateral and vertical diffusion above the source, the concentration at $z = h$ starts to fall whilst its value at the ground continues to increase, rising to a maximum considerably in excess of that previously experienced at $z = h$. Eventually, as in the case of the line-source (see § 1), the three profiles asymptotically approach one another for large x.

Finally figure 10 shows the effect of the shear when the eddy diffusivity is kept constant. For one curve the velocity shear is constant, conforming with the solution $\alpha = 1$, while the other curve represents the solution for $u =$ constant. The effect of the shear is summarized thus:

(i) the maximum is increased;
(ii) the maximum occurs earlier;
(iii) the maximum is 'peakier';
(iv) the rate of decrease of C with x for large x is unaltered.

8. Moving point-source

The idea behind the previous sections can be carried over and extended when the source is moving with constant velocity v in the transverse y-direction (figure 11). The source may be elevated at height h or at ground-level, since the reciprocal theorem of § 5 is still applicable. The solution

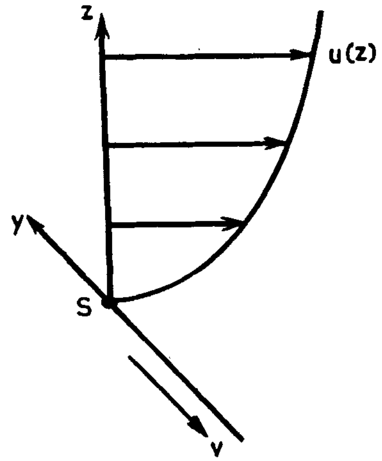

Figure 11. The source is moving with velocity v along the ground in a direction at right-angles to the unidirectional velocity field $u(z)$.

will therefore be sought for the ground-level source. It is found convenient to work with axes moving with the source so that the problem is that of a stationary point source in a wind field $(u(z), v, 0)$.

With u, K_z and K_y following (1.1), (1.4) and (1.5) and by the transformation of variables (4.1) with $K_1 = K_0$, the diffusion equation is

$$z^{\alpha}\frac{\partial C}{\partial x}+\frac{v}{K_0}\frac{\partial C}{\partial y}=\frac{\partial}{\partial z}\left[z^{1-\alpha}\frac{\partial C}{\partial z}\right]+z^{1-\alpha}\frac{\partial^2 C}{\partial y^2}. \tag{8.1}$$

This equation can be used to find the exact solutions for three functions of the concentration. As before, we may determine C_0 and C_2 ((1.7) and (1.8)), and since the distribution is no longer symmetric in the y-direction the third function is chosen to be

$$C_1 = \int_{-\infty}^{\infty} yC\,dy. \tag{8.2}$$

Now if $y = Y(x, z)$ denotes the lateral displacement of the centre of the plume then

$$C_1^* = \int_{-\infty}^{\infty} \{y - Y(x, z)\}C\,dy = 0. \tag{8.3}$$

Thus
$$C_1 = \int_{-\infty}^{\infty} yC\,dy = Y(x, z)\int_{-\infty}^{\infty} C\,dy = Y(x, z)C_0. \tag{8.4}$$

Therefore
$$Y(x, z) = C_1/C_0. \tag{8.5}$$

Equation (8.4) gives the physical interpretation of the function C_1. The function C_2 is still connected with the spread of the plume.

The spread is equal to C_2^*/C_0, where

$$C_2^* = \int_{-\infty}^{\infty} \{y - Y(x, z)\}^2 C\,dy. \tag{8.6}$$

Thus
$$\frac{C_2^*}{C_0} = \frac{C_2}{C_0} - Y^2(x, z). \tag{8.7}$$

The concentration may be represented by an expression equivalent to (1.9):

$$C = X(x, z)\exp - [\{y - Y(x, z)\}^2/f(x, z)], \tag{8.8}$$

where
$$X(x, z) = C_0\sqrt{\left(\frac{C_0}{2\pi C_2^*}\right)}, \qquad f(x, z) = 2\frac{C_2^*}{C_0}. \tag{8.9}$$

The first function C_0 is unaffected by v and therefore is given by (4.3), (4.4) and (4.6):

$$C_0 = \Lambda x^{-a}e^{-\eta}, \qquad a = \frac{1+\alpha}{1+2\alpha}, \qquad \eta = \frac{z^{1+2\alpha}}{(1+2\alpha)^2 x}. \tag{8.10}$$

The function C_1 satisfies the differential equation

$$\frac{\partial}{\partial z}\left[z^{1-\alpha}\frac{\partial C_1}{\partial z}\right] - z^{\alpha}\frac{\partial C_1}{\partial x} = -\frac{v}{K_0}C_0. \tag{8.11}$$

Putting

$$C_1 = \frac{v}{K_0}\Lambda(1+2\alpha)^{a-2}e^{-\eta}\mathscr{C}(\eta), \tag{8.12}$$

equation (8.11) becomes

$$\frac{\partial^2\mathscr{C}}{\partial\eta^2} + \left(\frac{a}{\eta} - 1\right)\frac{\partial\mathscr{C}}{\partial\eta} - \frac{a}{\eta}\mathscr{C} = -\eta^{a-2}, \tag{8.13}$$

the solution of which is

$$\mathscr{C}(\eta) = Ae^{\eta} + Be^{\eta}\int_0^{\eta}\eta^{-a}e^{-\eta}d\eta + \frac{e^{\mu}}{1-2a}\int_0^{\eta}\eta^{a-1}e^{-\eta}\,d\eta. \tag{8.14}$$

Now C_1 must satisfy the no-flux condition at the ground: $\eta^a(\partial C_1/\partial\eta) = 0$. Thus

$$B = 0. \tag{8.15}$$

Also $C_1 \to 0$ as $\eta \to \infty$, thus

$$A = \frac{1}{2a-1}\int_0^\infty \eta^{a-1}e^{-\eta}\,d\eta. \tag{8.16}$$

Whereby

$$C_1 = \Pi \int_\eta^\infty \eta^{a-1}e^{-\eta}\,d\eta, \tag{8.17}$$

where

$$\Pi = \frac{v}{K_0}\frac{\Lambda}{(2a-1)}(1+2\alpha)^{a-2}. \tag{8.18}$$

So that for large x, or for $z = 0$ (reverting to the real coordinates),

$$Y(x,z) \sim \frac{v}{u_0^\alpha K_0^{1-\alpha}}\frac{(1+2\alpha)^{a-2}}{(2a-1)}(a-1)!\,x^a. \tag{8.19}$$

The shape of $Y(x,z)$ on $z = 0$ for $\alpha = 0, \frac{1}{7}, \frac{1}{5}, \frac{1}{2}, 1$ are plotted in figure 12.

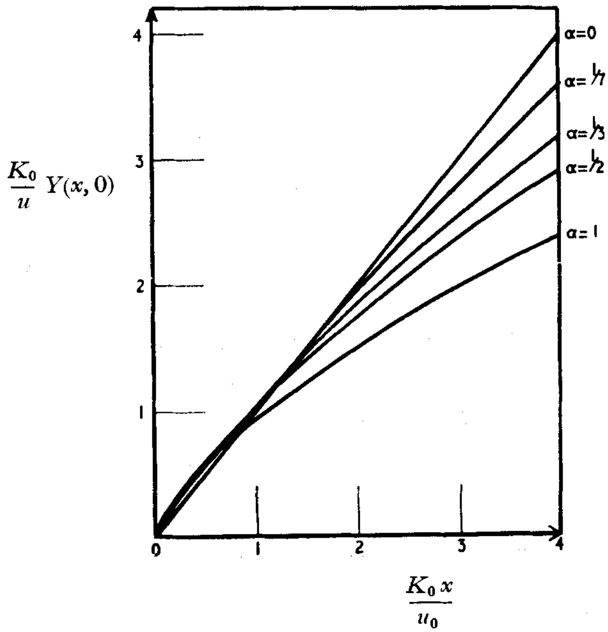

Figure 12. The displacement $Y(x, 0)$ of the centre of the plume at ground level, relative to the position of the moving source.

The third function is rather more involved due to the effect of the velocity v. However, we may write $C_2 = C_{2,1} + C_{2,2}$ where $C_{2,1}$ is the same as (4.20), in the real coordinates, and does not involve v. $C_{2,2}$ is the perturbation term and is found by a method very similar to that used in §4. It satisfies the differential equation

$$z^\alpha \frac{\partial C_{2,2}}{\partial x} = \frac{\partial}{\partial z}\left(z^{1-\alpha}\frac{\partial C_{2,2}}{\partial z}\right) + \frac{2v}{K_0}C_1 \tag{8.20}$$

in the variables (4.1).

Putting
$$C_{2,2} = 2\frac{v}{K_0}\Pi(1+2\alpha)^{2a-2}x^{a}e^{-\eta}\chi(\eta), \tag{8.21}$$
then (8.20) becomes
$$\frac{\partial^2\chi}{\partial\eta^2} + \left(\frac{a}{\eta}-1\right)\frac{\partial\chi}{\partial\eta} - 2\frac{a}{\eta}\chi = -\eta^{a-2}e^{\eta}\int_{\eta}^{\infty}\eta^{a-1}e^{-\eta}\,d\eta. \tag{8.22}$$
The solution, satisfying the zero-flux condition on $\eta = 0$, is
$$\chi = \frac{(2a-1)!}{(a-1)!}\Big[U(2a;a;\eta)\int_0^{\eta}\eta^{2a-2}{}_1F_1(2a;a;\eta)\int_{\eta}^{\infty}t^{a-1}e^{-t}\,dt d\eta -$$
$$-{}_1F_1(2a;a;\eta)\Big\{B+\int_0^{\eta}\eta^{2a-2}U(2a;a;\eta)\int_{\eta}^{\infty}t^{a-1}e^{-t}\,dt d\eta\Big\}\Big]. \tag{8.23}$$
B must be determined from the condition that $C_2 \to 0$ as $\eta \to \infty$. By a similar process as in §4,
$$B = -\frac{(2a-2)!\,(3a-2)!}{a(4a-2)!}. \tag{8.24}$$
The particular integral in (8.23) may be simplified by integrating by parts. It is found desirable to express $U(2a;a;\eta)$ in terms of the corresponding ${}_1F_1$-functions as before:
$$\text{P.I.} = \frac{1}{a-1}I^*(\eta,a-1)\Big[\eta^{1-a}{}_1F_1(1+a;2-a;\eta)\int_0^{\eta}\eta^{2a-2}{}_1F_1(2a;a;\eta)\,d\eta -$$
$$-{}_1F_1(2a;a;\eta)\int_0^{\eta}\eta^{a-1}{}_1F_1(1+a;2-a;\eta)\,d\eta\Big] +$$
$$+\eta^{1-a}{}_1F_1(1+a;2-a;\eta)\int_0^{\eta}\eta^{a-1}e^{-\eta}\int^{\eta}t^{2a-2}{}_1F_1(2a;a;t)\,dt d\eta -$$
$$-{}_1F_1(2a;a;\eta)\int_0^{\eta}\eta^{a-1}e^{-\eta}\int^{\eta}t^{a-1}{}_1F_1(1+a;2-a;t)\,dt d\eta. \tag{8.25}$$
Fortunately this can be simplified a great deal. Consider the first two terms. Integrating term by term and summing, we find that these terms reduce to
$$\frac{\eta^{a}e^{\eta}}{a(2a-1)}I^*(\eta,a-1) \tag{8.26}$$
(see (4.26) for I^*). The other two terms also simplify and can be expressed in terms of the function $V(b;a;\eta)$ defined in (4.19). Integrating term by term and collecting like powers of η, we finally obtain
$$\chi(\eta) = \frac{(2a-1)!\,(3a-2)!}{a(4a-2)!}\Big[\frac{(2a-2)!}{(a-1)!}\,{}_1F_1(2a;a;\eta) - \eta^{2a}V(2a;a;\eta)\Big] -$$
$$-\frac{\eta^{a}e^{\eta}}{a(2a-1)}I^*(\eta,a-1), \tag{8.27}$$
from which $C_{2,2}$ may be obtained using (8.21) and thus, with the use of (4.20), C_2. It should be noted that $C_{2,2}$ is proportional to v^2. This perturbation term increases the spread over the stationary-source case, and for large x this increase is proportional to x^{2a} (also on $z = 0$).

9. Conclusions

The main conclusion is that this approach has led to a fair measure of success in tackling the general problem of an elevated point-source. The most important results are represented by the equations (2.13), (3.8), (4.20), (5.6) and (6.23) and the graphs. Between them, they give an almost complete picture of the plume under varying conditions. The reciprocal theorem of § 5 has proved highly useful, but it should be pointed out in warning that this theorem only holds when the form of the eddy-diffusivity coefficients are such as have been taken in this paper (namely, independent of the source position) and cannot always be applied to solutions found under different assumptions, such as those of Davies. However, this is in no way serious, since, as has been indicated in the Introduction, the form of K used in this paper is considered to be theoretically preferable in the region in which the K-theory can be applied.

I should like to express my sincere gratitude to Professor M. J. Lighthill for his continual guidance and the numerous suggestions and fundamental ideas that have gone into the theory and the construction of this paper.

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